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LETTER TO THE EDITOR

'Twisted' Clebsch–Gordan coefficients for $SU_q(2)$

Ya I Granovskii and A S Zhedanov

Physics Department, Donetsk University, Donetsk 340055, Ukraine

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Abstract. A new approach to the Clebsch–Gordan problem for $SU_q(2)$ is proposed. The 'twisted' Clebsch–Gordan coefficients (TCGC) are defined to be the overlaps between 'twisted' connected and unconnected bases (being the q-analogues of rotated bases in standard SU(2)). The explicit expression for TCGC is found in terms of basic hypergeometric function ${}_4\Phi_3$ (Racah q-polynomials).

The quantum algebra $SU_q(2)$ is intrinsically anisotropic as is evident from its commutation relations

$$[j_0, j_\pm] = \pm j_\pm \qquad [j_+, j_-] = (\sinh 2\omega j_0)/\sinh\omega. \tag{1}$$

This property is closely related to its origin from the symmetry of the anisotropic XXZ spin chains ([1, 2] and others).

But then it signifies that the usual and somewhat trivial choice of the axis for quantization becomes important, so the previous freedom in that choice (as occurs for SU(2)) is lost. It means that exchange of the quantization axis (or more generally its rotation) might completely destroy the smooth rotation-invariant formulae.

These considerations are intimately related to the Clebsch–Gordan problem dealing with the correlation between two bases—the 'unconnected'

$$\psi_{j_1m_1j_2m_2} = \varphi_{j_1m_1} \otimes \varphi_{j_2m_2}$$

and the 'connected' Φ_{JM} one:

$$\Phi_{JM} = \sum_{m_1m_2} C(JM; j_1m_1j_2m_2)\psi_{j_1m_1j_2m_2} \tag{2}$$

because all the projections m_1, m_2, M are supposed taken on *the same* axis.

In the standard SU(2) algebra this assumption leads to no troubles: changing the axis is compensated by a unitary transformation of the states. Moreover this change may be different for ψ and Φ, while $C(JM; j_1m_1j_2m_2)$ itself remains unchanged.

What is the state of affairs in the $SU_q(2)$ case?

We may choose the basis $\psi_{j_1m_1j_2m_2}$ in standard form

$$\begin{aligned} j_0^{(i)}\psi_{j_1m_1j_2m_2} &= m_i\psi_{j_1m_1j_2m_2} \\ \hat{\kappa}_i\psi_{j_1m_1j_2m_2} &= k_i\psi_{j_1m_1j_2m_2} \end{aligned} \qquad i = 1, 2 \tag{3}$$

where

$$\hat{\kappa} = j_+j_- + \frac{\cosh(\omega(2j_0 - 1))}{2\sinh^2\omega}$$

is the Casimir operator for $SU_q(2)$ with eigenvalues

$$k_i = \frac{\cosh(\omega(2j_i+1))}{2\sinh^2\omega}$$

and the dimension of the representation is equal to $2j+1$.

The 'connected' basis Φ_{JM} is defined by the relations

$$J_0\Phi_{JM} = M\Phi_{JM} = (m_1+m_2)\Phi_{JM} \tag{4a}$$

$$\hat{\kappa}_J\Phi_{JM} = k_J\Phi_{JM} \tag{4b}$$

where $J_0, J_\pm$ are the generators of $SU_q(2)$ being the 'sum' of the initial ones [3]:

$$J_0 = j_0^{(1)} + j_0^{(2)} \qquad J_\pm = j_\pm^{(1)}\exp(\omega j_0^{(2)}) + j_\pm^{(2)}\exp(-\omega j_0^{(1)}). \tag{5}$$

The formula (2) then yields the 'longitudinal' Clebsch–Gordan coefficients (LCGC) for $SU_q(2)$ (the term 'longitudinal' stresses that the generators $j_0^{(i)}$ are diagonal).

The 'twisted' basis $\tilde{\psi}_{j_1m_1M}$ is defined to be the eigenstate for K_1 and L:

$$\begin{aligned} K_1\tilde{\psi}_{j_1m_1M} &= \lambda_{m_1}\tilde{\psi}_{j_1m_1M} \\ L\tilde{\psi}_{j_1m_1M} &= \lambda_M\tilde{\psi}_{j_1m_1M} \\ \hat{\kappa}_1\tilde{\psi}_{j_1m_1M} &= k_1\tilde{\psi}_{j_1m_1M} \end{aligned} \tag{6}$$

where

$$K_1 = j_-^{(1)}f(j_0^{(1)}) + f^*(j_0^{(1)})j_+^{(1)} + g(j_0^{(1)}) \tag{7a}$$

and

$$L = J_-F(J_0) + F^*(J_0)J_+ + G(J_0) \tag{7b}$$

are rotated ('twisted') generators.

The ansatz (7) is a generalization of the rotation in standard SU(2):

$$U(\theta)j_0U^+(\theta) = \sin\theta(j_+ + j_-)/2 + j_0\cos\theta \tag{8}$$

with coefficients $\cos\theta$ and $\sin\theta$ replaced by the functions $f(j_0)$ and $g(j_0)$.

The 'twist' of the 'connected' basis $\tilde{\Phi}_{JM}$ is defined by another pair of the relations

$$K_2\tilde{\Phi}_{JM} = k_J\tilde{\Phi}_{JM} \qquad L\tilde{\Phi}_{JM} = \lambda_M\tilde{\Phi}_{JM} \tag{9}$$

where K_2 is the full Casimir operator

$$K_2 = \hat{\kappa}_J. \tag{10}$$

For the schemes (6) and (9) to be compatible it is necessary for the operator L to commute with both K_1 and K_2:

$$[L, K_1] = 0 \tag{11a}$$

$$[L, K_2] = 0. \tag{11b}$$

Equation (11*b*) is evident by the definition of the Casimir operator K_2, but the relation (11*a*) should be verified: substituting (7) into (11*a*) one deduces that the only choice for K_1 and L (to within affine transformations) is

$$K_1 = aj_-^{(1)}\exp(\omega j_0^{(1)}) + a^*\exp(\omega j_0^{(1)})j_+^{(1)} + b\exp(2\omega j_0^{(1)}) \tag{12a}$$

$$L = aJ_-\exp(\omega J_0) + a^*\exp(\omega J_0) + b\exp(2\omega J_0) \tag{12b}$$

where a (b) are arbitrary complex (real) parameters.

The 'twisted' CGC (TCGC) are defined as

$$\tilde{\Phi}_{JM} = \sum_{m_1} \tilde{C}(JM; j_1 m_1) \tilde{\psi}_{j_1 m_1 M}. \tag{13}$$

In order to calculate TCGC $\tilde{C}(JM; j_1 m_1)$ it is sufficient to notice that the operators K_1, K_2 with their 'q-mutators' form the algebra

$$\begin{aligned} &[K_1, K_2]_\omega = K_3 \\ &[K_2, K_3]_\omega = BK_2 + C_1 K_1 + D_1 \\ &[K_3, K_1]_\omega = BK_1 + C_2 K_2 + D_2 \end{aligned} \tag{14}$$

where

$$[K_i, K_k]_\omega = \mathrm{e}^{\omega} K_i K_k - \mathrm{e}^{-\omega} K_k K_i \tag{15}$$

is the 'q-mutator'. The structure coefficients of the algebra (14) are given by

$$\begin{aligned} &B = 4 \sinh^2 \omega (bk_2 + k_1 \lambda_M) \\ &C_1 = \coth^2 \omega \qquad C_2 = -4|a|^2 \mathrm{e}^{\omega} \cosh^2 \omega \\ &D_1 = -2 \cosh \omega (bk_1 + k_2 \lambda_M) \\ &D_2 = 2 \cosh \omega (4|a|^2 \mathrm{e}^{\omega} \sinh^2 \omega\, k_1 k_2 - b\lambda_M) \end{aligned} \tag{16}$$

where k_i are taken from (3) and λ_M from (6) (due to operator L commutes with K_i it may be replaced by the constant λ_M).

The algebra with commutation relations (14) is known as the Askey–Wilson algebra AW(3), [4, 5].

In order to construct TCGC we need the value of the Casimir operator $\hat{Q}$ for AW(3) [5]:

$$\begin{aligned} \hat{Q} = \{K_3, \tilde{K}_3\}/2 + \cosh 2\omega (C_1 K_1^2 + C_2 K_2^2) \\ + B\{K_1, K_2\} + 2\cosh^2 \omega (D_1 K_1 + D_2 K_2) \end{aligned} \tag{17}$$

where $\tilde{K}_3 = [K_1, K_2]_{-\omega}$ and $\{\cdots\}$ denotes an anticommutator. Given the realization (10)-(12), the Casimir operator takes the value

$$\begin{aligned} Q = -|a|^2 \mathrm{e}^{\omega} \coth^4 \omega + 4|a|^2 \mathrm{e}^{\omega} \cosh^2 \omega (k_1^2 + k_2^2) \\ + 4 \sinh^2 \omega (b^2 k_2^2 + k_1^2 \lambda_M^2) - 8 \sinh^2 \omega \cosh 2\omega\, k_1 k_2 \lambda_M b \\ - \coth^2 \omega (b^2 + \lambda_M^2). \end{aligned} \tag{18}$$

Let us parametrize a and b by

$$a = v \qquad b = 2v^2 \sinh 2\omega t \tag{19}$$

where

$$v = \exp(\omega/2)/2 \sinh \omega. \tag{20}$$

(It is evident that only the ratio $b/|a|$ is essential for the TCGC problem.)

Then we obtain the spectra

$$\lambda_{m_1} = 2v^2 \sinh 2\omega (m_1 + t) \qquad \lambda_M = 2v^2 \sinh 2\omega (M + t) \tag{21}$$

$$-j_1 \leqslant m_1 \leqslant j_1 \qquad -J \leqslant M \leqslant J. \tag{22}$$

The main feature of AW(3) is that the overlap coefficients between the eigenstates of K_1 and K_2 are expressed in terms of the Askey–Wilson polynomials [6] (or basic hypergeometric function ${}_4\Phi_3$). Applying the technique of [4, 5] to our case we immediately obtain the explicit expression for TCGC:

$$\tilde{C}(JM; j_1 m_1) = C_0(x) h_n \, {}_4\Phi_3\left(\begin{matrix} q^{-n}, q^{-x}, q^{-2j_1-2j_2+x-1}, -q^{2t-2j_1+n} \\ q^{-N}, q^{-2j_1}, -q^{2t+M-j_1-j_2} \end{matrix} ; q \,\middle|\, q\right) \tag{23}$$

where the polynomials' parameters are

$$q = \exp(-2\omega) \qquad n = m_1 + j_1 \qquad x = j_1 + j_2 - J$$
$$N = M + j_1 + j_2. \tag{24}$$

The restrictions are assumed to be

$$0 \leqslant n, x \leqslant N \qquad M \leqslant j_2 - j_1 \leqslant 0 \qquad \omega > 0. \tag{25}$$

$C_0(x)$ and h_n are the weight amplitude and normalization factor being expressed in terms of Askey–Wilson polynomials' parameters [5, 6].

Thus, TCGC (23) do not coincide with LCGC: the latter are known to express via more simple ${}_3\Phi_2$ functions (Hahn q-polynomials) instead of ${}_4\Phi_3$ (Racah) ones (23) (for explicit expressions of LCGC in terms of Hahn q-polynomials see [7, 8]).

The LCGC can be obtained from TCGC by the limiting procedure $b \to \infty$. Indeed, in this limit the operator K_1 (12a) becomes $\exp(2\omega j_0^{(1)})$ and corresponding eigenstates $\tilde{\psi}_{j_1 m_1 M}$ coincide with 'longitudinal' ones (3). The functions ${}_4\Phi_3$ in (23) in the limit $b \to \infty$ (i.e. $t \to \infty$) are transformed into ${}_3\Phi_2$ due to $|q| < 1$. So, in this limit we indeed arrive at the 'ordinary' LCGC.

It is worth mentioning that in contrast to LCGC the operator L (12b) cannot be presented as a sum of two commuting operators belonging to the spaces where the independent momenta act (like $j_0^{(1)}$ and $j_0^{(2)}$ for the operator M in (4a)). So the quantum number m_2 is not defined for the 'twisted' problem, as indicated in our notation for basis: $\tilde{\psi}_{j_1 m_1 M}$ instead of $\psi_{j_1 m_1 j_2 m_2}$. In particular, the operator L cannot be obtained from J_0 by a unitary transformation because these operators have essentially different spectra (cf (4a) and (21)).

Thus, in contrast to the standard SU(2), the 'anisotropic' nature of $SU_q(2)$ leads to a non-trivial Clebsch–Gordan problem essentially depending on the choice of the basis. The more detailed analysis of TCGC and their applications will be published elsewhere.

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